Auctions with a minimum requirement of bids

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Abstract

We show that in a second–price sealed–bid procurement auction with independent and private costs of production, participation costs and the requirement of a minimum number of bids for the good to be bought, the unique Bayesian equilibrium and rationalizable solution calls for all firms not to bid.

To avoid this result, the buyer can commit to subsidize in certain circumstances the losing bidders; the amount of the prizes determines a cut–off value, such that all firms and only firms whose cost of production does not exceed this value enter the auction. Alternatively, the buyer can use a stochastic auction, where the provider of the good is not always the firm that bids the lowest price.

Keywords: procurement auction, entry cost, endogenous participation, minimum requirement of bids, subsidy, stochastic auction

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1. Introduction

Procurement auctions are widely used by governments and private firms to purchase indivisible goods or services, since they enable the buyer to exploit the competition among the bidders and help prevent dishonest dealings between the buyer’s agent and the supplier.

In this paper, we focus on the second–price sealed–bid Vickrey auction with private and independent costs of production. An attractive property of this auction is that, for each firm, it is always optimal to enter the auction and bid the true cost of production, whatever the strategy of the other potential suppliers. In terms of formal analysis, this conclusion is straightforward; it is also intuitively plausible, because it only requires firms to understand that their bid only affects the probability of winning, and not the payment they receive in case they win.

In many procurement auctions, bidding is costly. In particular, bidders can incur participation costs, which are either costs of learning their costs of production, paid by \textit{ex–ante} identical firms (McAfee and McMillan, 1987, Engelbrecht–Wiggans, 1987, 1993, Levin and Smith, 1994) or costs of preparing the bid, borne after the costs of production are private information (Samuelson, 1985, Stegeman, 1996, Menezes and Monteiro, 1999). These costs are paid only by the firms that participate to the auction and, unlike the entry fees, do not accrue to the buyer of the good.

When participation costs are taken into account, some firms can choose not to bid; so, participation to auctions is endogenous; however, if a firm bids, it will optimally bid its cost of production.

In the present paper, we concentrate on costs of preparing a bid \textit{à la} Samuelson. In this setting, the unique symmetric Bayesian equilibrium calls for all firms and only firms whose cost of production does not exceed a cut–off value to enter the auction. This value is shown to be negatively related to the amount of the entry cost; at the limit, when this cost tends to zero, all firms whose cost of production does not exceed the ceiling price set by the buyer participate to the auction. With respect to the above result, in terms of formal analysis the present conclusion is less straightforward, because it does not necessarily follow from the assumption of common knowledge of Bayesian rationality; moreover, for this conclusion to hold, it is required
not only that an equilibrium is played, but also that the equilibrium is symmetric. However, also in this case the result is not particularly puzzling, in the sense that it is intuitively plausible that, when there are participation costs, firms with a high cost of production do not enter the auction, because they think they have a low probability of winning, whereas firms with a low cost of production bid. It is also intuitively plausible that some firms that in a given situation prefer not to bid would instead participate to the auction should the entry cost be sufficiently low.

In models where participation to the auction is endogenous, the buyer may have the incentive to influence the number of participants, for example by requiring a minimum number of bids (strictly greater than one) for the good to be bought. This requirement actually characterizes many auctions and may also be justified on completely different grounds. Many transactions take place between a buyer’s agent and the potential suppliers, and the buyer cannot know the number of firms to which the agent asked to make an offer, whereas the number of bids is observable at no costs. In this case, the buyer can set a minimum requirement of bids in order to reduce the risk of dishonest dealings between the agent and the supplier. Or the buyer can fear a collusive behavior of the potential suppliers; when there are very few competing firms and the main benefit from collusion are savings in preparation costs, the above requirement can induce even firms that intend to act cooperatively to make an offer, and this reduces their incentive to collude. Or the buyer cannot know the distribution of the costs of production of the potential suppliers; so, when there are few bids, the buyer is unable to evaluate the fairness of the offers and can decide to obtain additional information before buying the good. Finally, a government selling multiple licences can decide to provide no licences at all whenever there is only one offer, in order to avoid the making of a monopoly.

To the best of our knowledge, the introduction of a minimum requirement of bids in an auction procedure was never studied before. In the present paper, we show that in a second–price sealed–bid auction with a positive cost of preparing a bid this requirement always has the powerful effect to make the situation where all firms do not bid be the unique Bayesian equilibrium and
rationalizable solution of the game, whatever the amount of the entry cost, the number of potential bidders, the costs of production of the different firms and the hierarchy of beliefs over these costs. In terms of formal analysis, this conclusion is therefore straightforward: it is implied by the assumption of common knowledge of Bayesian rationality, and also arises when an equilibrium (not necessarily symmetric) is assumed to be played. However, it is somewhat “paradoxical”, in the sense that it is clearly at odds with the intuition that in many auctions firms are almost certain that other potential suppliers will bid and, hence, that the requirement of, for example, at least two bids for the good to be bought is empirically irrelevant. And it contrasts with the intuition that a firm with a low cost of production should participate to the auction, provided the entry cost is sufficiently low. So, one is tempted to consider the no–bid result an inadequate description of the way firms actually behave or should behave. This would call for him to abandon the canonical auction model with independent and private values or to question the validity of the game–theoretic concepts of solution. Alternatively, one can ask how the buyer can avoid the no–bid result. In this paper, we show that the buyer can subsidize in certain circumstances bidders that do not win the auction; the amount of the prizes determines a cut–off value such that, as in Samuelson (1985), the unique Bayesian equilibrium calls for all firms and only firms whose cost of production does not exceed that value to enter the auction. Or the buyer can make sure that the provider of the good is not always the firm that bids the lowest price.

The structure of the paper is the following. In section 2, we briefly analyze the model described in Samuelson (1985) in case a second–price sealed–bid auction is used. In section 3, we introduce the minimum requirement of offers and prove the no–bid result, whereas in sections 4 and 5 we show how monetary prizes or a stochastic auction can lead back to an equilibrium characterized by a cut–off value. Section 6 provides some concluding remarks.
2. Entry costs and endogenous participation

An individual has decided to buy an indivisible good through a second–price sealed–bid Vickrey auction.\(^1\) As usual, we model this problem as a game with incomplete information. In particular, we assume that there are \(n\) potential suppliers, and let the private cost of production of each firm be independently distributed in the interval \([c, \bar{c}] \subset \mathbb{R}^+\) according to a commonly known, atomless distribution function \(F(c)\). We also assume that preparing a bid costs \(k\) dollars; this cost is paid after the firms have learned their costs of production. To simplify the notation, we assume that the bidders are risk–neutral and, to allow for the existence of bids, we let \(c_0 > \bar{c} + k\), where \(c_0\) is the buyer’s valuation of the good. Finally, we assume that if only one firm makes a bid, it will provide the good at the ceiling price \(s\), optimally chosen by the buyer.\(^2\)

This problem was initially studied in Samuelson (1985) in the context of a first–price sealed–bid auction. In this section, we show that a second–price auction leads to the same conclusion in terms of the types of firms that participate to the auction.\(^3\)

If we define \(c^* \in [c, \bar{c}), c^* < s\) as follows, it is straightforward to see that \(c^*\) exists and is unique.

**Definition 1.** \([1 - F(c^*)]^{n-1}(s - c^*) - k = 0.\)

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\(^1\) Stegeman (1996) show that, in the model we consider in this section, this auction has desiderable efficiency properties.

\(^2\) The opportunity for the buyer to set a ceiling price below its valuation of the good is well known since Laffont and Maskin (1980). However, when participation costs are taken into account, this conclusion is not so clear–cut. For example, McAfee and McMillan (1987) and Levin and Smith (1994) show that with a positive cost of learning the private cost of production, it is never optimal for the buyer to set such a ceiling price. In the model we consider in this section, we will show that it is optimal for the buyer to set \(s \neq c_0\) whenever there are no entry fees.

\(^3\) It also gives rise to the same expected payment for the buyer of the good. For insights on this equivalence, see Stegeman (1996) and Menezes and Monteiro (1999). In these papers, a sale auction was considered, and a symmetric conclusion was obtained; this implies that our analysis can easily be rephrased in terms of this type of auction. We refer henceforth to a procurement auction, both in the model and in references to other literature.
To interpret this definition, let us consider a firm whose cost of production is $c$, and assume for a moment that all firms with a lower cost of production make a bid, whereas firms with a higher cost do not. This implies that the firm under consideration obtains the good only when it is the unique bidder.\footnote{To be precise, the marginal bidder can be asked to provide the good also when it faces bidders that have the same cost of production. But this occurs with probability zero; moreover, the firm earns a zero profit. Henceforth, we neglect this case.} In a second–price auction, if the firm participates, the usual reasoning based on the notion of dominance calls for this firm to bid its cost of production; as a consequence, its expected profit in entering the auction is $[1 - F(c)]^{n-1}(s - c) - k$. So $c^*$ is the maximum $c$ such that $[1 - F(c)]^{n-1}(s - c) - k \geq 0$, that is, the maximum cost of production that induces a firm to participate to the auction.

Now, let us remove the above assumption on the behavior of the firms and concentrate on the symmetric equilibria of the game.\footnote{This is absolutely standard in auction theory. To the best of our knowledge, the only exception to the use of an equilibrium concept in auctions is Battigalli and Siniscalchi (2000), where the authors investigate whether the usual results in first–price auctions can be obtained on the basis of rationality conditions alone. As to the symmetry assumption, see Stegeman (1996) for interesting insights on the model we consider in this section.} The following Proposition shows that $c^*$ still works as a cut–off value.

**Proposition 1.** The unique symmetric Bayesian equilibrium calls for all firms and only firms whose cost of production does not exceed $c^*$ to make a bid.

**Proof.**

Let $\sigma(c)$ be the probability that, in equilibrium, a firm whose cost of production is $c$ makes a bid and $\hat{c}$ be the cost of a firm which bids with positive probability. From the definition of equilibrium, this firm must have a non–negative expected profit. As a consequence, given the assumption of symmetry, if a firm with a lower cost bids, it will have a strictly positive expected profit. Therefore, $\sigma(\hat{c}) > 0 \Rightarrow \sigma(c) = 1$ for all $c < \hat{c}$. This implies that if there exists a symmetric Bayesian equilibrium where some (but not all) types of firms whose cost
of production does not exceed the ceiling price enter the auction — let us call it an interior equilibrium — this is characterized by a cut–off value, such that all firms and only firms whose cost of production does not exceed this value make a bid. In this equilibrium, the marginal bidder is indifferent between entering the auction or not; this implies that its cost of production $c$ is such that $[1 - F(c)]^{n-1}(s - c) - k = 0$. But this is the definition of $c^*$, which is known to exist and be unique. This proves that an interior equilibrium exists and is unique. As for boundary equilibria, the situation where all firms do not enter the auction is not an equilibrium, because a firm whose cost of production is $c$ could earn $s - c - k > 0$ by entering the auction.\(^6\) Also the situation where all firms whose cost of production does not exceed the ceiling price enter the auction is not an equilibrium, because a firm whose cost is $c = s$ would suffer a loss equal to the entry cost $k$; so, it would prefer not to bid. This proves that the interior equilibrium characterized by a cut–off value $c^* \in \left[\underline{c}, \bar{c}\right]$, $c^* < s$ is the unique equilibrium of the game. □

If one asks whether this conclusion is an adequate representation of the way firms actually behave or should behave, the answer need not be negative. Indeed, it seems intuitively plausible that a firm with a low cost of production participates to the auction, despite the entry cost, because it thinks it has a high probability of winning the auction, whereas a firm with a high cost of production does not. It is also intuitively plausible that some firms that in a given situation prefer not to bid would instead participate to the auction should the entry cost be sufficiently low. However, the existence of a bidding firm, the marginal bidder, that never wins the auction against other bidders is clearly at odds with the intuition that a firms participates to an auction because it hopes it will win the competition of other actual (and not only potential) bidders.\(^7\)

\(^{6}\) If the ceiling price $s$ was such that $s - \underline{c} - k \leq 0$, the buyer would obtain the good with probability zero and, hence, the auction would be completely useless for it. Given the assumption $c_0 \geq \underline{c} + k$, the above ceiling price would imply the buyer to forgo a profitable opportunity; as a consequence, such a ceiling price could not be optimal. Henceforth, we focus on ceiling prices $s$ such that $s - \underline{c} - k > 0$.

\(^{7}\) This property of the equilibrium is not peculiar to Samuelson (1985), but also characterizes the Bayesian equilibrium of the canonical auction model with private and independent costs of production, no entry costs and a
In this model, the number of bidders is a random variable, and the buyer may have the incentive to affect the distribution of this variable. Samuelson (1985) and Menezes and Monteiro (1999) show that the expected payment for the buyer need not be increasing in the number of potential bidders \( n \); so, the buyer may have the incentive to limit the number of potential bidders. But the buyer may also affect the expected number of bidders through a ceiling price that differs from its valuation of the good or an entry fee.

Let \( f > 0 \) be an entry fee paid to the buyer of the good; if \( f < 0 \), we have a subsidy. A bidder treats the entry fee and the costs of preparing the bid in the same way; as a consequence, the cut-off value \( c^* \) is determined by the equation

\[
[1 - F(c^*)]^{n-1}(s - c^*) - (k + f) = 0
\]

It is straightforward to see that \( c^* \) is increasing in \( s \) and decreasing in \( f \); so, the buyer can choose the equilibrium cut-off value by appropriately setting \( s \) and \( r \).

If the buyer raises the entry fee or lowers the ceiling price, a trade-off arises: on the one hand, an increase in the entry fee raises the revenue the buyer obtains from the fees and a lower ceiling price implies a lower expected payment for the buyer, conditional on the good being bought; on the other hand, because of the effects on the cut-off value, they decrease the probability that the good will be bought. Menezes and Monteiro (1999) pointed out that the strategy that minimizes the expected payment for the buyer depends on the distribution function \( F(c) \), and demonstrated that it is never optimal for the buyer to set \( s = c_0 \) and \( f = 0 \). In fact, they proved that if the buyer sets a ceiling price equal to its valuation of the good, it will optimally charge a positive entry fee, but this is not to say that it is strictly optimal for the buyer to introduce an entry fee, as we show in the following proposition.

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8 Samuelson (1985) and Stegeman (1996) also analyze the effects of a change in the number of potential bidders on the expected social cost of the auction.
Proposition 2. Let \( s \) and \( f \) be, respectively, a ceiling price and an entry fee, and let \( \tilde{c} \) be the corresponding cut–off value. All the pairs \( (s, f) \) that give rise to the same cut–off value \( \tilde{c} \) imply the same expected payment for the buyer.

Proof.

Let \((ds, df)\) be a change in the ceiling price and in the entry fee that give rise to the same cut–off value \( \tilde{c} \); hence,

\[
[1 - F(\tilde{c})]^{n-1}ds - df = 0
\]

The change in the expected payment for the buyer is

\[
nF(\tilde{c})df - \binom{n}{1} F(\tilde{c})[1 - F(\tilde{c})]^{n-1}ds
\]

which, taking into account the above relation between \( ds \) and \( df \), is equal to zero. ■

According to this proposition, for any given cut–off value and, hence, for any given equilibrium, the buyer can reach that equilibrium by setting \( s = c_0 \) and \( f \neq 0 \), or \( s \neq c_0 \) and \( f = 0 \), or \( s \neq c_0 \) and \( f \neq 0 \), but they all have the same expected cost. So, we can neglect the entry fee; in this case, from the above result by Menezes and Monteiro, it follows that it is optimal for the buyer to set \( s \neq c_0 \).

Of course, the buyer can affect the number of bidders in many other ways; for example, it can require a minimum number of bids (strictly greater than one) for the good to be bought. The next section will show that this requirement leads to a rather disappointing conclusion.

3. The requirement of a minimum number of bids

Now, let us modify the model and assume that the good is not bought when there is only one bid; in what follows, we can leave unspecified whether the good is bought or not when there are two or more bids.

Assumption 1: The good is not bought when there is one bid.
This assumption is clearly satisfied when the buyer requires a minimum number of bids $m > 1$ for the good to be bought. This requirement is indeed rather common in practice. For example, when an employee is asked to buy a good on behalf of his firm, he is generally required to gather a minimum number of offers. This can easily be explained within a traditional principal–agent model: if the principal cannot know (or it is too costly to verify) the number of firms to which the agent asked to make an offer, whereas the actual number of bids is observable at no costs, the minimum requirement of offers seems to be the obvious way to reduce the risk of dishonest dealings between the agent and the supplier of the good. As a second example, consider a firm that has asked a given number of potential suppliers to make an offer. If the firm has never bought that service before, the assumption that it knows the distribution of the costs of production is problematic. And if one lets the buyer have only beliefs over this distribution, the offers provide this firm useful information over the true distribution. So, a firm receiving few bids is not able to evaluate the fairness of the offers and can decide to obtain additional information before buying the service.\(^9\) As a final example, consider a government selling licenses through a multi–unit auction. The government exploits the competition between the potential suppliers in order to maximize its revenue. This can induce the government to commit to sell a number of licenses that is strictly lower than the number of firms participating to the auction;\(^10\) an implication of this requirement is that no licenses are sold when there is only one bid.\(^11\)

\(^9\) As an additional example where the existence of multiple offers is valuable for the buyer, consider a procurement auction for the provision of a good like a theater, a bridge or a stadium. In general, the characteristics of these goods are not wholly determined by the auction procedure, but depend in some way on the projects that are presented. In such a context, it is not surprising that the buyer commits not to buy the good whenever there are too few offers.

\(^10\) Just to quote an example, in the Italian five licenses Spectrum Auction it was explicitly stated that “if the number of offers is lower or equal to five, in order to guarantee an effective competition between the bidders, . . . , the number of licenses can be reduced by the Comitato dei Ministri, . . . , in such a way that the number of licenses sold is one less than the number of offers”.

\(^11\) We think that this can better be explained by the government’s desire to avoid the making of a monopoly.
The following proposition shows that Assumption 1 has a powerful effect on the equilibrium of the game.

**Proposition 3.** *The unique Bayesian equilibrium calls for all firms not to bid.*

**Proof.**

Let us initially observe that, as usual, for each bidder it is always optimal to bid its cost of production. Let now $H_i \subseteq [\bar{c}; \tilde{c}]$ be the set of types for firm $i$ such that, in equilibrium, this firm bids with positive probability, and define $H = \bigcup_{i=1}^{n} H_i$. If $H \neq \emptyset$, let $h = \sup_{c} H$. Any firm whose cost of production lies in the interval $(h - k, h]$ will have a negative expected profit. This implies that, in equilibrium, this firm will not bid. But this contradicts the definition of $h = \sup_{c} H$. Therefore, $H = \emptyset$, that is, in equilibrium all types of firms do not bid. ■

To better understand this Proposition, let us come back to the equilibrium described in the previous section. In that equilibrium, the marginal bidder — that is, the firm whose cost of production was $c^*$ — was never able to win the auction against another firm, and earned a positive profit only when it was the unique bidder. In terms of equilibrium analysis, Assumption 1 implies that now the marginal bidder cannot have a positive profit. Indeed, either it will be the unique bidder or will be under–bid. In both cases, it will have to pay the entry cost and will not provide the good. It follows that, in equilibrium, this bidder cannot exist, that is, no firm participates to the auction.

Now, let us show that the same conclusion also obtains as unique rationalizable solution of the game. This implies that the no–bid result follows from the assumption of common knowledge of Bayesian rationality.

**Proposition 4.** *The unique rationalizable solution calls for all firms not to bid.*

**Proof.**

Let us consider a firm whose cost of production lies in the interval $(\tilde{c} - k, \tilde{c}]$. This firm knows
that, if it bids, it will have a negative expected return, whatever the strategies of the other potential suppliers. Since it is Bayesian rational, it will choose not to bid. From the assumption of common knowledge of Bayesian rationality (henceforth, CKR), also this conclusion will be common knowledge. Let us consider now a firm whose cost lies in the interval \((\bar{c} - 2k, \bar{c} - k]\). This firm knows that it will never be able to win the auction against a firm whose cost of production exceeds \(\bar{c} - k\). Therefore, it knows that, if it bids, it will have a negative expected profit. Since it is Bayesian rational, it will choose not to bid. From the assumption of CKR, also this conclusion will be common knowledge. And so on. It follows that, for each \(M \in \mathbb{N}\) (where \(\mathbb{N}\) is the set of positive integers), it will be common knowledge that a firm whose cost of production lies in the interval \((\bar{c} - Mk, \bar{c} - (M - 1)k]\) will not bid. Given the arbitrariness of \(M\), all firms must not bid, whatever their cost of production.

In terms of formal analysis, the no–bid result is straightforward, because it is implied both by the assumption of common knowledge of Bayesian rationality and by the notion of Bayesian equilibrium. However, it seems rather “paradoxical”, also because it is completely independent of the number of potential suppliers, of their costs of production, of their beliefs over the costs of production of the other firms, of the amount of the entry cost and of the ceiling price set by the buyer. In particular, the discontinuity in the types of firms participating to the auction with respect to the amount of the entry cost stretches our intuition: for any \(k > 0\), all firms do not participate to the auction, whereas when \(k = 0\) making a bid is a weakly dominant strategy for all firms whose cost of production does not exceed the ceiling price; this justifies the assumption that these firms enter the auction. So, one can find the no–bid conclusion an inadequate representation of the way bidders behave or should behave. This would call for him to abandon the canonical auction model with independent and private values or to question the validity of the game–theoretic concepts of solution. Though we find both approaches challenging and interesting, in the present paper they will not be further investigated. Alternatively, one can ask how the buyer can avoid the no–bid result. This will be the object of the next two sections.
4. Auctions, reimbursements and subsidies

In this section, we continue to assume that the good is bought through a second–price sealed–bid auction, and ask how the buyer can avoid the no–bid result. A straightforward implication of the above reasoning is that the marginal bidder can exist only if it is given in certain circumstances a sort of reward for participating to the auction. This implies that, in some situations, the buyer must give a prize to the losing bidders.

In general, the buyer can make the situations where a prize is paid and the amount of the prize depend on the number and value of the different bids, and this can affect the strategies of the potential suppliers in two different ways: on the one hand, by influencing the decision to participate to the auction and, on the other hand, by affecting the bidding strategy of the firms that participate.

To simplify the analysis, in this paper we focus on prizes whose amount depends on the number of the actual bids, but not on their value,\(^\text{12}\) so that it is still optimal for all the firms that enter the auction to bid their cost of production.

In this setting, the more general reward scheme, \(\{\pi_i, z_i\}_{i=1,2,...,n}\), specifies for any number \(i\) of bids, \(i = 1, 2, \ldots, n\), the probability \(\pi_i\) that each bidder obtains a payment \(z_i\),\(^\text{13}\) We look for the Bayesian equilibria of the game.

**Proposition 5.**

Let \(p_i^B(c) = \binom{n-1}{i-1} [F(c)]^{i-1} [1-F(c)]^{n-i}, \quad B(c) = \sum_{i=1}^{n} p_i^B(c) \pi_i z_i - k\) and 
\(C = \{c \in (\bar{c}, \bar{e}) \mid c < s |B(c) = 0\}\).

(a) If \(C \neq \emptyset\), for any \(c \in C\) the situation where all firms and only firms whose cost of production

\(^{12}\) By so doing, we rule out the cases where the buyer tries to incentivize particular bids. This is rather common in the so–called beauty contests, where the buyer often gives a consolation prize to a limited number of losers in order to improve the quality of the bids. Riley and Samuelson (1981) showed that, in the traditional model with private and independent values, all these auctions, which they call Santa Claus Auctions, are optimal.

\(^{13}\) We do not require this value to be non–negative; so, we also allow for the existence of entry fees.
does not exceed \( c \) enter the auction is a symmetric Bayesian equilibrium.

(b) If \( B(c) \leq 0 \), there exists a symmetric Bayesian equilibrium where all firms do not enter the auction. This is the unique symmetric Bayesian equilibrium if and only if \( B(c) < 0 \) for all \( c \in [c, \bar{c}], \ c \leq s \).

(c) If \( B(s) \geq 0 \), there exists a symmetric Bayesian equilibrium where all firms whose cost of production does not exceed the ceiling price enter the auction. This is the unique symmetric Bayesian equilibrium if and only if \( B(c) > 0 \) for all \( c \in [c, \bar{c}], \ c < s \).

(d) No other symmetric Bayesian equilibria exist.

(e) A symmetric Bayesian equilibrium always exists.

**Proof.**

The first part of the proof of Proposition 1 showed that, in a second–price sealed–bid auction with entry costs, if there exists a symmetric Bayesian equilibrium where some (but not all) firms whose cost of production does not exceed the ceiling price enter the auction, the equilibrium is characterized by a cut–off value, such that all firms and only firms whose cost of production does not exceed this value bid. This is true even when there is a minimum requirement of bids for the good to be bought. It follows that the only other possible symmetric equilibria of the game are the situation where all firms whose cost of production does not exceed the ceiling price enter the auction and the situation where all firms do not bid. This proves part (d) of the proposition.

Now, if we assume that all firms and only firms whose cost of production does not exceed \( c \leq s \) make a bid, for each bidder the probability that it is one out of \( i \) bidders is

\[
p_i^B(c) = \binom{n-1}{i-1} [F(c)]^{i-1} [1 - F(c)]^{n-i}
\]

A firm with a cost of production \( c \) never provides the good; hence, its expected profit is

\[
B(c) = \sum_{i=1}^{n} p_i^B(c) \pi_i z_i - k
\]

Let us focus now on the equilibria characterized by an interior cut–off value \( c \in (c, \bar{c}), \ c < s \). If \( B(c) < 0 \), it is not optimal for a firm whose cost of production is \( c \) to enter the auction; as a
consequence, the situation under consideration is not an equilibrium. And if \( B(c) > 0 \), also firms with a cost of production which exceeds \( c \) can profitably enter the auction. So, in an interior equilibrium the marginal bidder must have an expected profit equal to zero. This implies that \( c \) must be such that \( B(c) = 0 \). This is an equation of degree \( n - 1 \) in \( F(c) \). So, there may exist multiple values of \( c \) that solve this equation. Each of them gives rise to a symmetric Bayesian equilibrium, where all firms and only firms whose cost of production does not exceed this value enter the auction. Indeed, firms with a lower cost of production have a positive expected profit, since they obtain the same prizes as the marginal bidder but, in addition, they sometimes win the auction, whereas firms with a higher cost of production get the same expected profit as the marginal bidder and, hence, are indifferent between entering the auction or not. This proves part (a) of the Proposition.

Let us consider now the case where \( B(c) \leq 0 \). This implies that, if all firms do not enter the auction, a firm that bids obtains a non-positive expected profit. Hence, not bidding for all firms is a Bayesian equilibrium. If \( B(c) < 0 \) for all \( c \in [c, \bar{c}] \), \( c \leq s \), then \( C = \emptyset \); so, according to point (a), there exist no interior equilibria and, hence, not bidding is the unique symmetric Bayesian equilibrium. This proves the if-part of (b).

As for the case where \( B(s) \geq 0 \), this inequality says that, given that all firms whose cost of production does not exceed the ceiling price enter the auction, a firm that bids obtains a non-negative expected profit. So, always bidding is a symmetric Bayesian equilibrium. If \( B(c) > 0 \) for all \( c \in [c, \bar{c}] \), \( c < s \), then \( C = \emptyset \); so, according to point (a), there exist no interior equilibria and, hence, always bidding is the unique symmetric Bayesian equilibrium. This proves the if-part of (c).

The only-if parts of (b) and (c) follow immediately from the continuity of \( B(c) \) and from the intermediate value theorem. A necessary condition for an equilibrium where all firms do not enter the auction is \( B(c) \leq 0 \). If \( B(c) = 0 \), then \( C \neq \emptyset \), and there also exists an equilibrium where firms with a cost of production \( c \) bid. So, a necessary condition for a unique no-bid equilibrium
is $B(c) < 0$. Now, either $B(c) < 0$ for all $c \in [c, \hat{c}]$, $c \leq s$ and, hence, this equilibrium is unique, or there exists $\hat{c}$ such that $B(\hat{c}) \geq 0$. But if $B(\hat{c}) = 0$, then $C \neq \emptyset$ and, from (a), there also exists an interior equilibrium. Finally, if $B(\hat{c}) > 0$, $B(c)$ is a continuous function, with $B(c) < 0$ and $B(\hat{c}) > 0$. So, according to the intermediate value theorem, there exists $\hat{c} \in (c, \hat{c})$ such that $B(\hat{c}) = 0$. But then $C \neq \emptyset$ and, from (a), there also exists an interior equilibrium. This proves the only–if part of (b). An analogous proof works for the only–if part of (c).

Finally, either $C \neq \emptyset$ and, hence, from (a) there exists an interior equilibrium, or $C = \emptyset$. In the latter case, from the intermediate value theorem, either $B(c) < 0$ for all $c \in [c, \hat{c}]$, $c \leq s$ or $B(c) > 0$ for all $c \in [c, \hat{c})$, $c < s$. In both cases, from (b) and (c), a symmetric equilibrium exists. This proves part (e) of the proposition.

The above proposition gives insights on different types of reward schemes.

Let us initially consider the case where the buyer gives no prizes at all, that is, $\pi_i z_i = 0$ for all $i = 1, 2, \ldots, n$. We have $B(c) = -k < 0$ for all $c \in [c, \hat{c}]$. So, according to Proposition 5, the unique symmetric equilibrium calls for all firms not to bid, as we already know from section 3.

As a second example, let us consider the case where the buyer always reimburses the entry cost, whatever the number of bids, that is, $\pi_i = 1$ and $z_i = k$ for all $i = 1, 2, \ldots, n$. In this case, $B(c) = 0$ for all $c \in [c, \hat{c}]$, $c \leq s$. Hence, according to Proposition 5, for any $c \in [c, \hat{c}]$, $c \leq s$, the situation where the firms whose cost of production does not exceed $c$ enter the auction is a symmetric Bayesian equilibrium. So, there exists a continuum of equilibria.\(^\text{14}\)

As a third example, let us consider the case where the buyer reimburses the entry cost only whenever the good is not bought. In this case, $\pi_i z_i$ is either $k$ or (with positive probability) $0$. This implies that $B(c) < 0$ for all $c \in [c, \hat{c}]$, $c \leq s$. So, according to Proposition 5, the unique symmetric Bayesian equilibrium calls for all firms not to bid.

\(^{14}\) Note however that if the buyer always reimburses the entry cost, making a bid is a weakly dominant strategy.

So, one can reasonably assume that all firms whose cost of production does not exceed the ceiling price enter the auction.
The above proposition, as it stands, raises two problems. On the one hand, given a symmetric equilibrium, this can generally be obtained using different reward schemes; so, one can ask what is the reward scheme that is less costly for the buyer. On the other hand, when there exist multiple equilibria, there is the problem of selection between equilibria. In the remaining part of the section, we will aim at solving these problems.

Let us assume for a moment that, when there are multiple equilibria, the buyer can choose the equilibrium it prefers. Let us further suppose that, for some reason, the buyer has decided that the marginal bidder must have a cost \( c^{**} \) such that the situation where all firms and only firms whose cost of production does not exceed \( c^{**} \) enter the auction is a Bayesian equilibrium of the game. The following proposition provides an interesting neutrality result.\(^{15}\)

**Proposition 6.** Let \( R \) be the set of reward schemes such that the situation where all firms and only firms whose cost of production does not exceed \( c^{**} \) enter the auction is a Bayesian equilibrium of the game. In this equilibrium, all the reward schemes in \( R \) have a cost \( k n F(c^{**}) \) for the buyer.

**Proof.**

In equilibrium, the probability that in the auction there will be \( i \) bids is

\[ p_i(c^{**}) = \binom{n}{i} [F(c^{**})]^i [1 - F(c^{**})]^{n-i} \]

The buyer chooses the reward scheme \( \{\pi_i, z_i\}_{i=1,2,\ldots,n} \) that minimizes its expected payment

\[ \sum_{i=1}^{n} p_i(c^{**}) i \pi_i z_i, \text{ under the constraint } \]

\[ B(c^{**}) = \sum_{i=1}^{n} p_i^B(c^{**}) \pi_i z_i - k = \frac{1}{n F(c^{**})} \sum_{i=1}^{n} p_i(c^{**}) i \pi_i z_i - k = 0 \]

\(^{15}\) This conclusion is reminiscent of the revenue equivalence theorem, initially discussed in Vickrey (1961) and further investigated in Myerson (1981) and Riley and Samuelson (1981). However, now the firm which makes the lowest bid does not always provide the good, because of the minimum requirement of bids; so, we cannot rely on the canonical version of the theorem.
that is, the expected profit for the marginal bidder $c^{**}$ must be equal to zero.

From the constraint, we have $\sum_{i=1}^{n} p_i(c^{**}) i \pi_i z_i = k n F(c^{**})$. So, the cost for the buyer of reaching the equilibrium characterized by a cut-off value $c^{**}$ is always $k n F(c^{**})$, whatever the reward scheme.\footnote{Note that this is the result of two countervailing forces, which exactly compensate. On the one hand, for a given prize and a given probability that there will be $i$ bidders, the buyer prefers to pay the prize when the number of bids is low, because the prize is paid (with a given probability) to all bidders. This is illustrated by the term $i$ in the sum $\sum_{i=1}^{n} p_i(c^{**}) i \pi_i z_i$, the buyer minimizes. On the other hand, we have $\frac{p^B_i(c^{**})}{p_i(c^{**})} = \frac{i}{n F(c^{**})}$. So, the effect of the conditional probability gives the buyer the incentive to pay the prize when the number $i$ of bids is high.}

According to this proposition, given a Bayesian equilibrium, the buyer can choose \textit{any} reward scheme that leads to that equilibrium, because all these schemes have the same cost. This implies that, for any cut-off value $c^{**}$, we can concentrate on \textit{one} particular reward scheme that makes the situation where all firms and only firms whose cost of production does not exceed $c^{**}$ enter the auction be a symmetric equilibrium of the game. To this purpose, let $\pi_i z_i = 0, i = 2, 3, \ldots, n$, that is, let the buyer pay a prize only when there is one bid. In this case, the zero expected profit condition for the marginal bidder, $B(c^{**}) = 0$, simplifies to $[1 - F(c^{**})]^{n-1} \pi_1 z_1 - k = 0$. The expected reward the buyer pays whenever there is one bid, $\pi_1 z_1$, and the cut-off value, $c^{**}$, are negatively related. This implies that, for \textit{any} given expected reward $\pi_1 z_1$, there exists a unique symmetric Bayesian equilibrium $c^{**}$. With this reward scheme, we are therefore able to rule out the problem of selection between multiple equilibria; as a consequence, the assumption that the buyer is able to choose the equilibrium it prefers is not further needed. Moreover, for \textit{any} cut-off value $c^{**} \in (c; \bar{c}), c^{**} < s$, there exists an expected reward $\pi_1 z_1$ such that the unique equilibrium calls for all firms and only firms whose cost of production does not exceed $c^{**}$ to enter the auction.

So, the buyer’s problem of choosing the equilibrium cut-off value can be solved by focusing on this simple reward scheme. And the optimal expected reward $\pi_1 z_1$ is the solution to a trade-off: on the one hand, the negative effect of a higher prize on the bidder’s payment when it faces one
bidder; on the other hand, the positive effect of a higher prize on the cut–off value, which implies a lower probability to pay the prize and a higher probability to buy the good.

5. Stochastic auctions and beauty contests

In this section, we continue to assume that a firm buying a good asks several potential suppliers to make an offer, preparing a bid is costly and the buyer requires a minimum number of offers \( m \) for the good to be bought, but modify the auction procedure.

The first part of the proof of Proposition 1 demonstrated that, whatever the procedure that determines the provider of the good, if there exists a symmetric Bayesian equilibrium where some (but not all) firms whose cost of production does not exceed the ceiling price enter the auction, this must be characterized by a cut–off value such that all firms and only firms whose cost of production does not exceed this value bid. Without prizes, the marginal bidder can have a positive expected return that offsets the participation cost only if, in some circumstances, it is given the opportunity to provide the good at a price that exceeds its cost of production; it follows that, when \( m > 1 \), to avoid the no–bid result, the good must not always be provided by the firm that bids the lowest price.\(^{17}\) This occurs in practice in the so–called beauty contests, where the buyer maintains full discretionality in choosing the provider of the good, or in the auctions where the buyer commits to choose between the two, three, \ldots, firms that bid the lowest price. In these cases, features other than the value of the bids play a crucial role in determining the winner of the auction and the price that will be paid.

The procedure used by the buyer can affect both the firms’ decision to enter the auction and their bidding strategy. In this section, we concentrate on a very peculiar procedure, that is meant to represent only an example showing the importance of a stochastic rule in avoiding the no–bid result, and that has the desirable property that, if a firm enters the auction, it will optimally bid its cost of production; the similarities with the analysis of the previous section allow us to

\(^{17}\) This type of auction is known as stochastic auction. See Riley and Samuelson (1981, p.389).
maintain the same notation and to characterize the symmetric equilibria of the game without an explicit proof.

We assume that, when there is a number $i$ of bids that exceeds the minimum requirement $m$, the good is provided with probability $1 - \pi_i$ by the firm that bids the lowest price at the second lowest price, as in the traditional second-price sealed-bid auction. However, now there is a probability $\pi_i$ that the winner is chosen at random, irrespective of the values of the bids; in this case, the price is $z_i$. So, when there are $i \geq m$ bids, a firm whose cost of production is $c$ has a probability $\pi_i / i$ of having a profit $z_i - c$; this term is a sort of prize, which is given with positive probability to any bidder. As a consequence, the analysis runs exactly as in section 4, the only difference being that now the value of the prize is not the same for all types of firms, but is decreasing in the cost of production.

**Proposition 7.**

Let $B(c) = \sum_{i \geq m} p_i^B(c) \frac{\pi_i}{i} (z_i - c) - k$ and $C = \{ c \in (c, \bar{c}), c < s | B(c) = 0 \}$.

(a) If $C \neq \emptyset$, for any $c \in C$ the situation where all firms and only firms whose cost of production does not exceed $c$ enter the auction is a symmetric Bayesian equilibrium.

(b) If $B(c) \leq 0$, there exists a symmetric Bayesian equilibrium where all firms do not enter the auction. This is the unique symmetric Bayesian equilibrium if and only if $B(c) < 0$ for all $c \in [c, \bar{c}], c \leq s$.

(c) If $B(s) \geq 0$, there exists a symmetric Bayesian equilibrium where all firms whose cost of production does not exceed the ceiling price enter the auction. This is the unique symmetric Bayesian equilibrium if and only if $B(c) > 0$ for all $c \in [c, \bar{c}), c < s$.

(d) No other symmetric Bayesian equilibria exist.

(e) A symmetric Bayesian equilibrium always exists.

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18 This price can, but need not, be equal to the ceiling price $s$. Note that Samuelson’s model obtains as a special case, with $m = 1, \pi_1 = 1, z_1 = s$ and $\pi_i = 0, i = 2, 3, \ldots, n$. 

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6. Conclusion

We showed that in a second-price sealed-bid procurement auction with independent and private costs of production and positive costs of preparing a bid, the introduction of a minimum requirement of bids for the good to be bought makes the situation where all firms do not bid be the unique Bayesian equilibrium and rationalizable solution of the game. This requirement is theoretically meaningful and actually characterizes many auctions; so, it was worth investigating when and how the buyer can avoid the no-bid result. We showed that a first escape route is to give in certain circumstances a subsidy to bidders that do not win the auction, a sort of consolation prize. We concentrated on the reward schemes where the prizes depend on the number of bids, but not on the value, so that it is still optimal for each firm that enters the auction to bid its cost of production, and showed that, by appropriately choosing the amount of these prizes, the buyer is able to choose a cut-off value, such that all firms and only firms whose cost of production does not exceed that value enter the auction. We also proved that all the reward schemes that lead to the same equilibrium have the same cost for the buyer. Finally, we showed that a second way out of the no-bid result is to make sure that the winner is not always the firm that bids the lowest price. This can occur in practice when the buyer maintains some degree of discretionality in choosing the provider of the good, as in the so called beauty contests or, more generally, when features other than the value of the bids play a role in determining the provider of the good.
References


