The investment-uncertainty relationship, risk-aversion and optimal leverage

[REVISED, OCTOBER 2004]

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Abstract
We develop a partial equilibrium dynamic model in which firms are risk-averse. We analyse the determinants of the investment-uncertainty relationship by means of numerical techniques. When firms can borrow “outside” resources at the riskless rate, an increase in price volatility depresses investment for realistic parameter values. In our model, portfolio considerations play an important role. When the marginal revenue of capital becomes more uncertain, the risk-averse firm’s owners reduce their “short position” in the risk-free asset, thus diminishing the firm’s debt level. The contraction in leverage reduces the expected returns on investment because the expected marginal revenue product is higher than the user cost of capital. In turn, the reduction in expected returns tends to depress investment.

Keywords: Investment, uncertainty, risk-aversion, leverage.

JEL classification codes: D92, E22

* I am indebted to Gianmaria Martini and Domenico Delli Gatti for their comments and suggestions. I have also benefited from many suggestions from the referees. Usual disclaimers apply. Financial support from C.N.R. and M.U.R.S.T. has been greatly appreciated.
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1. Introduction

Economic theory offers several alternative explanations about the relationship between uncertainty and firms’ investment. Models that assume that capital cannot be freely adjusted while the other inputs are flexible typically suggest that investment is positively related to uncertainty. A second stream of literature focuses on irreversible investments, usually finding that they are reduced by an increase in price uncertainty. Finally, models that take into account risk aversion in firm ownership have not yet yielded clear-cut results.

We consider a framework where a risk-averse competitive firm can optimally choose its leverage ratio, and we show that price uncertainty tends to depress the equilibrium rate of investment.

A central feature of our paper is that uncertainty cannot be diversified away by firms’ owners; hence, our analysis applies to small or medium-sized firms. These businesses are often seriously limited in their opportunities to gain access to the equity market: they are frequently owned by a few agents and represent a large share of these agents’ wealth. This may happen because the owners directly manage the firm, which means that selling shares of the company to outside financiers could induce moral hazard problems. Thus, the owners must often invest a large share of their own wealth in the business.\(^1\) Asymmetric information issues may also play an important role because high-quality entrepreneurs sometimes find it convenient to finance a large share of their firms’ capital to signal their merit. Also, uninformed potential financiers may prefer to loan funds to firms through banks instead of purchasing equities. The lack of diversification provides a strong motivation for taking into account firm owners’ risk aversion. It should be noted that in our model we will abstract from the possibility that firm owners partly diversify their portfolio through the equity market, since this

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\(^1\) The “deed of partnership” usually constrains (and often denies) the right to sell shares to non-members, which would introduce new partners into the firm.
assumption would act in favour of our results.\(^2\)

The firms' access to external finance is the second important ingredient of our model. We believe that it is important to incorporate the leverage choice into the analysis of investment decisions under uncertainty because the debt-to-internal resource ratio is significant in almost every advanced economy. In particular, in continental Europe and in Japan the debt-to-equity ratio typically varies between two and four, and it ranges between 0.5 and 1 even in the Anglo-Saxon countries, where firms' access to equity markets is wider.\(^3\) To analyse these issues, we adopt a partial equilibrium framework. We cannot draw upon a general equilibrium-representative agent model because that would imply that the debt in the risk-free asset must be zero in net supply. In other words, the use of a general equilibrium-heterogeneous agents model, involving relevant analytical difficulties—and the need to heavily rely on numerical techniques—would tend to obscure our points.

Following many classic papers (see, e.g., Hartman (1972) and Abel (1983)), we consider a situation in which the capital level must be decided upon before the shock is observed, while some other productive input, usually referred to as “labour,” can be freely adjusted after the realisation of the stochastic process. Within this framework, the standard result is that a competitive firm should expand its investment in response to an increase in the output price volatility. It is easy to provide an intuitive explanation for this result: the asymmetry in the timing of decisions implies that the firm’s marginal revenue of capital is convex in the price shock.\(^4\) Hence, a mean preserving spread in the shock increases the expected marginal revenue of capital and therefore the firm’s investment activity.

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\(^2\) Using a Merton-type CAPM, one can show that the “portfolio shift” toward safer assets decreases both the investment in a now-riskier business and the optimal leverage for the firm.

\(^3\) See Davis (1992) for a thorough assessment of financing behaviour and financial structures, which also provides a comparative assessment of many advanced economies.

\(^4\) In fact, when the firm can adjust the employment of labour once it has observed the realisation of price, the increase in the marginal revenue following a positive shock is magnified by the expansion in output due to an increase in the flexible factor employment. In contrast, when the realisation is unfavourable, the firm reduces its labour input so that the contraction in the marginal revenue is moderate.
We find that this result is reversed when risk-averse firm owners are free to optimally choose the firm’s leverage ratio. In our model, as in Craine (1989) and Zeira (1990), a portfolio shift effect plays an important role: when the marginal revenue of capital becomes more uncertain, the risk-averse firm’s owners reduce their short position in the risk-free asset, thereby diminishing the firm’s debt level. The contraction in leverage reduces the expected returns on investment because the expected marginal revenue product is higher than the user cost of capital (the firm’s ownership bears a risk premium). In addition, the reduction in expected returns tends to depress savings and investment.

We believe our result is interesting because the empirical evidence is in favour of a negative relationship between investment and uncertainty (see, e.g., Ferderer (1993), Leahy and Whited (1996) and Guiso and Parigi (1999)). In the latter paper, it is shown that the relation is negative—although scarcely significant—for firms characterised by low degrees market power). Caselli et al. (2003), using sectoral data, document that demand uncertainty had negative effects on investment in Europe during the nineties.

After a capsule review of the literature on this issue, we present our model in Section 2. Even in the simplest framework, in which the utility function belongs to the constant relative risk aversion family and the arguments of the Cobb-Douglas production function are labour and capital, it is not possible to obtain a closed-form solution for the model. Thus, we develop a numerical routine, and we “calibrate” the model, finding, in Section 3, that a permanent increase in volatility depresses the equilibrium average investment when firms’ owners are strongly risk-averse.5

In Section 4, we present two developments of our basic framework. Again exploiting numerical techniques, we first adopt a CES technology. We focus on cases in which the constant elasticity of substitution between capital and labour is lower than unity, since this seems appropriate for describing short-period situations, and we find

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5 As we shall see, the choice to analyse the equilibrium investment level tends to favour the attainment of a positive “investment-uncertainty” relationship.
that the “investment-uncertainty” relationship is negative even with moderate degrees of risk aversion for the firm. Finally, we revert to a three-factor Cobb-Douglas technology in which labour and capital must be chosen in advance, while “raw materials” (or unskilled labour) remain flexible. Here, we find again a negative uncertainty-investment relationship for realistic values of the risk-aversion parameter. Our conclusions are presented in Section 5.

As already remarked, several important contributions, such as Oi (1961) and Hartman (1972), highlight that a mean preserving spread in the price shock increases the expected marginal revenue of capital and therefore the firm’s investment activity. As underscored by Abel (1983) and others, the analysis extends to the case in which capital can be adjusted at a convex cost after the realisation of the shock.

The curvature of the marginal revenue function also plays an important role in Caballero (1991). He shows that the positive relation between investment and uncertainty weakens as the price elasticity of the demand function faced by the firm decreases. Hence, the degree of competition of the environment in which firms operate becomes an important factor in explaining the sign of the investment-uncertainty relationship. In our model, we find a negative investment-uncertainty relationship without the need to refer to non-competitive market structures, which might be unappealing when explaining the behaviour of small or medium-sized firms. The empirical analysis by Guiso and Parigi (1999) supports the view that monopolistic power is important in explaining the reduction in investment caused by an increase in demand uncertainty.

An important stream of the literature has shown that, when investment is irreversible and indivisible, it is reduced by an increase in price uncertainty (see, e.g., Dixit (1989), Dixit and Pindyck (1994), and Bertola (1998)). While the marginal revenue product of capital is still typically convex in the output price, the persistence in

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6 To understand this result, notice that the lower the elasticity of demand, the less convex the marginal revenue function. In fact, the increase (reduction) in the flexible input following a positive (negative) shock reduces (increases) the output price. Moreover, Caballero shows that the “investment-uncertainty relation” may easily turn negative if one adds asymmetric adjustment costs to imperfect competition.
the stochastic process describing uncertainty generates a “value of waiting”—it is usually rational to sink the cost only when a sufficiently high level of expected revenues has been reached. The larger the variance in the output price, the higher is the trigger in expected revenues, and thus, on average, the longer the procrastination in making investment decisions. Guiso and Parigi (1999) provide support for the view that irreversibility is significant in accounting for the negative relationship between uncertainty and investment. However, they find that such a relationship is negative (although scarcely significant) even for firms characterised by low degrees of irreversibility.

The more recent literature focuses on imperfectly competitive markets, in which the delaying effect of the “value of waiting” may be undermined by the fear of being pre-empted (and temporarily excluded by the market) by competitors. For example, Weeds (2002) neatly analyses an oligopolistic model of investment in R&D along these lines.

However, this recent literature does not take into account a third possible explanation for the negative relationship between investment and price uncertainty—risk aversion in the firm ownership. While this idea has been given relatively little attention, it is certainly not new. In two early papers, Sandmo (1971) and Leland (1972) analysed factor demands under output price uncertainty; however, they assumed that all the inputs must be chosen before the output price is known. Hartman (1976) examined the role of risk-aversion in a model in which capital is the unique “quasi-fixed” factor and firms employ a decreasing-returns CES production function. His results were not clear-cut.

Craine (1989) sets up a general equilibrium portfolio model to assess whether the financial effect—namely the decrease in demand for an asset that becomes more risky—may overrun the expected profit effect highlighted by Hartman and Abel. With an intertemporal utility function logarithmic in consumption (and thus characterised by a low degree of risk-aversion), he finds that the capital optimally allocated to the firm
facing more uncertainty decreases only if the number of firms is low. In this case, in fact, every firm contributes significantly to aggregate uncertainty; hence, it is perceived as “very risky.” Accordingly, a mean-preserving spread in the firm’s technology induces a negative portfolio shift even in the presence of an increase in its expected returns.

The “portfolio adjustment effect” also plays an important role in Zeira (1990). He sets up a model in which risk-averse investors can choose between a safe technique and a risky one, and under plausible assumptions he finds that an increase in uncertainty (and returns, through the convexity effect) in the stochastic technology involves a shift in resources toward the safe one. However, in his model overall savings are fixed, and thus investment is exogenous, while in our framework it is endogenously determined.

Nakamura (1999) provides another contribution close to the present one. He sets up a dynamic partial equilibrium model in which the competitive firm is risk averse. He suggests that the price volatility-investment relationship is negative when the degree of risk aversion is lower than unity but higher than the labour productivity parameter in his Cobb-Douglas production function. His result can be criticised from different points of view: in particular, several studies have concluded that the degree of risk aversion is higher than 1. In contrast, our framework can account for a negative investment-uncertainty relationship in the presence of realistic degrees of risk aversion. More
recently, Kumar (2003), analyses a model in which the variance of a (persistent) technological shock is imperfectly known. He finds that, in a general equilibrium in which the representative agent is reasonably risk averse, the uncertainty concerning the variance of the shock moderately increases the equilibrium investment.

2. The model

In our model, as in many contributions, the competitive firm exploits a Cobb-Douglas production function: \( y_t = L_t^\alpha K_{t-1}^{1-\alpha} \). Notice that the period-\( t \) output (\( y_t \)) is obtained by means of the contemporaneous labour input (\( L_t \)) and of the previous-period capital stock (\( K_{t-1} \)), \( \alpha \in (0,1) \). This formulation is useful for underscoring the fact that, while the labour input can be decided upon once the realisation of the price process is known, capital must be present at the beginning of the production process. Both the wage (\( w \)) and the price of investment goods (\( q \)) are fixed. Part of the physical capital is funded directly out of the firm’s own resources (\( F_t \), which is expressed in terms of the capital goods), while the remainder is financed through debt (\( D_t \)), which means that \( K_t = F_t + D_t / q \). For ease of exposition, we choose \( q = 1 \). Debt must be remunerated at the riskless rate \( r \). The output price \( p_t \) is a serially independent random variable with mean \( \bar{p} \) and variance \( \sigma_p^2 \); the support for \( p_t \) is such that \( p_t \geq 0 \). Defining \( I_t \) as the investment level, the firm’s cash flow at time \( t \), \( \pi_t \), is:

\[
\pi_t = p_t L_t^\alpha K_{t-1}^{1-\alpha} + \Delta D_t - w L_t - r D_{t-1} - I_t,
\]

where \( \Delta D_t = D_t - D_{t-1} \). Because labour can be chosen when the price is known, substitution of the optimal value for the labour input in the above expression gives:

\[
\hat{\pi}_t = h p_t^\beta K_{t-1} + \Delta D_t - r D_{t-1} - I_t,
\]

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9 The stochastic structure for the price process is a relevant difference from Nakamura. He chooses a geometric Brownian motion specification for price, implying that the latter is non-stationary; whereas we opt for the polar case, assuming that the output price distribution is identical and independent over time. While this hypothesis is important for obtaining a closed-form solution for our model, we believe that it is sensible in a framework in which the cost of capital is fixed. Roughly speaking, while the “random-walk” assumption for a single price makes sense, if one considers a fixed price for capital as a normalisation our hypothesis provides a way to stylise the co-movement among prices.
where $\varepsilon = 1/(1-\alpha)$ and $h = (1-\alpha)(\alpha/w)\varepsilon$. Hence, $hp_t^\varepsilon$ is the marginal revenue product of capital.

The dynamics for the firm’s internal resources is:

$$F_t = F_{t-1} - \delta K_{t-1} + I_t - \Delta D_t,$$

where $\delta$ is the (constant) depreciation rate.

Firms are owned by risk-averse agents, whose utility function is specified as:

$$u(\hat{\pi}_t) = (1 - \gamma)^{-1} \hat{\pi}_t^{(1-\gamma)},$$

where $\gamma > 0$ represents both the degree of relative risk aversion and the reciprocal of the elasticity of intertemporal substitution.\(^{10}\) The maximum value function for the firm’s owners is:

$$J(p_t, F_{t-1}, D_{t-1}) = \max \{ F_t, D_t \} \sum_{i=0}^{\infty} E_t[\beta^i u(\hat{\pi}_{t+i})],$$

where $E_t[.]$ denotes expected values based on the information available at time $t$ and $\beta \in (0,1)$.

To underscore the fact that investment is funded partly by the increase in debt

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\(^{10}\) Following Epstein and Zin (1989) or Weil (1990), it is possible to distinguish risk aversion from elasticity of intertemporal substitution. One needs to assume that preferences can be written as the intertemporal objective:

$$U(\hat{\pi}_t, \hat{U}_{t+1}) = [\hat{\pi}_t - S + \beta \hat{U}_{t+1}]^{1/(1-S)},$$

where $\hat{U}_{t+1}$ is the certainty equivalence of utility at time $t + 1$; and $S \in [0,1)$ represents the reciprocal of the elasticity of intertemporal substitution. When the degree of relative risk aversion is constant,

$$\hat{U}_{t+1} = \{E_t[U_{t+1}^{(1-\gamma)}]\}^{1/(1-\gamma)},$$

and the indirect utility function takes the form:

$$J(p_t, F_{t-1}) = \max \{ F_t, D_t \} \sum_{i=0}^{\infty} E_t[\beta^i u(\hat{\pi}_{t+i})],$$

where $A = \left\{ 1 + \varepsilon \beta E_t[\psi_i^{(1-\gamma)} (1-\gamma)] \right\}^{-\varepsilon}$. We performed several numerical simulations using this alternative specification, and we found that the results that we shall present in the main text are not appreciably modified as long as $S > 1$. Because many empirical estimates suggest that the elasticity of intertemporal substitution is about one third (and hence $S \approx 3$), we chose not to adopt this more complex specification for the utility function.
and partly out of “internal” resources, we write:

\[ I_t = \delta (F_{t-1} + D_{t-1}) + \Delta D_t + \Delta F_t, \]  

(1)

and we express the internal resources devoted to investment as a linear function of \( F_{t-1} \):

\[ F_t = (1 + z_t) F_{t-1}. \]  

(2)

Moreover, it is convenient to express the firm’s optimal debt level as a proportion of the value of its (end-of-period) physical resources:\footnote{Specifying debt as a share of the beginning of period internal resources yields the same results at the price of a slightly more involved algebra.}

\[ D_t = \theta_t F_t. \]

Using the expression above and eqs. (1-2) to reformulate \( \hat{\pi}_t \), we can express the maximum value function for the representative firm as:

\[ J(p_t, F_{t-1}) = \max \{z_t, \theta_t\} \{ u(\hat{\pi}_t) + \beta E_t [J(p_{t+1}, F_t)] \}, \]  

(3)

where \( \hat{\pi}_t = [(hp_t^\varepsilon (1 + \theta_{t-1}) - (r + \delta)\theta_{t-1} - (z_t + \delta)]F_{t-1}. \)

We now solve the above problem under the following hypotheses:

**Assumption 1:** \( \gamma \geq 1 \)

**Assumption 2:** \( h\tilde{p}^\varepsilon \geq (r + \delta) > E_t[(hp_t^\varepsilon)^{1-\gamma}] / E_t[(hp_t^\varepsilon)^{\gamma}]. \)

Using Assumption 1, we focus on the empirically relevant case. In addition, this hypothesis guarantees the optimality of the solution we characterise in what follows. Because the marginal revenue function \( hp_t^\varepsilon \) is convex in output price, Assumption 2 implies that the expected marginal revenue product is higher than the user’s cost of capital (i.e., \( E_t[hp_{t+1}^{\varepsilon}] \geq (r + \delta) \)). The wedge between \( r + \delta \) and the expected marginal revenue represents the premium necessary to induce a risk-averse firm to contract a positive debt level. Moreover, Assumption 2 imposes a lower bound on \( r \), which will be exploited to guarantee the finiteness of the solution for \( \theta_t. \)\footnote{The inequality \( E_t[(hp_t^\varepsilon)^{1-\gamma}] / E_t[(hp_t^\varepsilon)^{\gamma}] < h\tilde{p}^\varepsilon \) can be verified by means of a Taylor’s approximation around \( \overline{p} \).}
The tentative solution we suggest for the problem (3) takes the form:

\[
J(p_t, F_{t+1}) = A \left[ 1 - \delta + hp_t^\xi + (hp_t^\xi - (r + \delta))\theta_{t-1} \right]^{(1-\gamma)} \frac{F_{t+1}^{1-\gamma}}{(1-\gamma)}
\]  

(4)

where \(A\) is an underdetermined coefficient, which must be positive, otherwise the maximum value function would not be increasing with the level of the firm’s resources.

The first order condition for the maximum value problem (3) with respect to \(\theta_t\) implies:

\[
E_t \left\{ \left[ (1 - \delta + hp_{t+1}^\xi + (hp_{t+1}^\xi - (r + \delta))\hat{\theta}_t \right] \right\} ^\gamma \left[ hp_{t+1}^\xi - (r + \delta) \right] = 0.
\]  

(5)

To characterise the solution, \(\hat{\theta}_t\), we now prove the following:

**Proposition 1:** When Assumptions 1 and 2 are satisfied, \(\hat{\theta}_t\) is unique and \(\hat{\theta}_t \in (-1, (1-\delta)/(r+\delta))\).

**Proof:** Refer to the Appendix.

\(\hat{\theta}_t\) depends upon the distribution of future prices because the firm can increase the physical capital installed at the end of period \(t\) by contracting debt in the same period, but such capital only becomes productive in period \(t+1\). The above equation clearly suggests that the optimal leverage depends upon output price volatility. Although it is not possible to solve eq. (5) analytically, a simple intuition suggests that an increase in future price volatility must reduce \(\hat{\theta}_t\): for a given \(\theta_t\), an increase in \(\sigma_p^2\) augments the covariance between the marginal value of the indirect utility function (the first term in eq. 5) and the marginal revenue product of capital. Thus, risk-averse firm owners reduce \(\hat{\theta}_t\) in order to reduce fluctuations in their indirect utility function. This intuition will be confirmed by our numerical simulations.

Notice also that \(\hat{\theta}_t\)—obtained by means of eq. (5)—is independent of the state variable \(F_t\) and \(A\), while it depends upon \(p_{t+1}\). Because equation (5) determines \(\hat{\theta}_t\) as a function of \(p_{t+1}\) and of the parameters, we exploit the second first-order condition to determine \(z_t\) for a given \(\hat{\theta}_t\):

\[
\begin{align*}
& hp_t^\xi (1 + \hat{\theta}_{t-1}) - (\delta + (r + \delta)\hat{\theta}_{t-1} - z_t) F_{t+1}^{1-\gamma} \\
& - \beta AE_t \left\{ \left[ (1 - \delta + hp_{t+1}^\xi + (hp_{t+1}^\xi - (r + \delta))\hat{\theta}_t \right] \right\}^{(1-\gamma)} (1 + z_t)^{-\gamma} F_{t+1}^{1-\gamma} = 0,
\end{align*}
\]  

(6)
where we have used eq. (2). Some algebra allows formulation of the share of the firm’s internal resources used for investment as:
\[
zt = \frac{\gamma \varepsilon \beta \theta \delta}{1 + \{\beta \varepsilon \theta \delta\}^{-1/\gamma}}
\]
where \( \psi_{t+1} \equiv \{1 - \delta + h p_{t+1} \varepsilon + [h p_{t+1} \varepsilon - (r + \delta)]\hat{\theta}_t\} \). Substituting (7) into the Bellman equation (3), we determine \( A \):
\[
A = \left(1 - \frac{1}{\{\beta \varepsilon \theta \delta\}^{-1/\gamma}}\right)^{-\gamma}.
\]

Notice that \( A \) is independent of time and state because \( p_{t+1} \) is i.i.d.; hence, our guess is verified. We now prove:

**Proposition 2:** When Assumptions 1 and 2 are satisfied, \( A \) is positive.

**Proof:** Refer to the Appendix.

Inserting (8) into (7), we obtain:
\[
zt = \frac{1 - \delta + h p_{t+1} \varepsilon + [h p_{t+1} \varepsilon - (r + \delta)]\hat{\theta}_t}{\{\beta \varepsilon \theta \delta\}^{-1/\gamma}} - 1.
\]

This completes the solution for the maximum value problem (3). Hence, we prove:

**Proposition 3:** When Assumptions 1 and 2 are satisfied, the tentative solution (4) attains the supremum for the maximum value problem (3) with \( A \) given by (8), and \( z_t \) given by (9).

**Proof:** Refer to the Appendix.

Because \( D_t = \hat{\theta}_t F_t \), the investment ratio can be expressed, from eq. (1), as:
\[
\frac{I_t}{K_{t-1}} = \delta + \hat{\theta}_t (1 + z_t) - \hat{\theta}_{t-1} + z_t.
\]

Notice that, if \( \theta \) were to be held arbitrarily constant—a case that we are about to consider—the net investment ratio would be simply given by \( z_t \).
From (9), we notice that $E_t[\psi_{t+1}^{1-\gamma}]$ and $z_t$ move together. To understand this point, substitute eq. (8) into (6) to get:

$$[h\rho^\varepsilon(1 + \hat{\theta}_{t-1}) - [\delta + (r + \delta) \hat{\theta}_{t-1}] - z_t]^{-\gamma} = [\beta E_t[\psi_{t+1}^{1-\gamma}]]^{-1/\gamma} - 1 \right]^{\gamma}(1 + z_t)^{-\gamma}.$$

This equation simply tells us that, given the debt level, investment is optimally set to balance the marginal utility of current cash flow with the (expected) marginal utility of a next-period increase in firms’ internal funds, a result that closely parallels the classic findings in the literature studying consumers’ intertemporal optimisation problems.

One immediately notices that the right-hand side of the equation above increases with $E_t[\psi_{t+1}^{1-\gamma}]$. Accordingly, investment is positively related to such expectation because an increase in the latter augments the expected marginal utility of firms’ internal funds.

We approximate the effect of a mean-preserving spread in future prices on investment by considering the second-order Taylor expansion of $E_t[\psi_{t+1}^{1-\gamma}]$ around $\bar{p}$, for a given $\theta_t$ (hence, $\theta_t = \theta_{t-1} = \bar{\theta} > -1$):

$$E_t[\psi_{t+1}^{1-\gamma}] \approx [1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta}]^{1-\gamma} +$$

$$+ (1 - \gamma)hp^\varepsilon\varepsilon (1 + \bar{\theta}) \{1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta} \}^{-\gamma} E_t[p_{t+1} - \bar{p}] +$$

$$+ (1 - \gamma)hp^\varepsilon\varepsilon^{-2} (1 + \bar{\theta}) \{1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta} \}^{-\gamma}$$

$$\left\{-\gamma \left[\frac{h\bar{p}^\varepsilon(1 + \bar{\theta})}{1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta}}\right] + (\varepsilon - 1)\right\} \frac{\sigma_p^2}{2}.$$

Since Assumption 2 guarantees that $\{1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta} \} > 0$, we have:

$$\text{Sign} \left\{ \frac{\partial z_t}{\partial \sigma_p^2} \right\} = \text{Sign} \left\{ \frac{\partial E_t[\psi_{t+1}^{1-\gamma}]}{\partial \sigma_p^2} \right\} =$$

$$= \text{Sign} \left\{ - (1 - \gamma) hp^\varepsilon(1 + \bar{\theta}) \left[1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta} \right] + (1 - \gamma)(\varepsilon - 1) \right\} \frac{\sigma_p^2}{2}.$$
Eq. (11) suggests that three effects influence the relationship between net investment and uncertainty. The first two have been known in the literature on precautionary savings since Sandmo (1970). A mean-preserving spread in the rate of return of future investment increases the expected marginal utility of future consumption, thereby inducing a tendency toward higher savings (and investment); however, an increase in savings would also positively affect the variance of future consumption, an undesirable effect for a risk-averse individual. As is well known, with CRRA preferences the former effect is stronger than the latter when $\gamma > 1$. These two effects boil down to the first addendum in the curly brackets in eq. (11). However, in our case, the convexity in the firm’s profit function implies that a mean preserving spread in the output price brings about an expansion in the expected return on capital. The increase in the future (expected) marginal revenue product of capital—making investment more productive—favours savings (via the “intertemporal substitution” effect), but it also involves an income effect, which plays in favour of an increase in current consumption. The second addendum in the curly brackets of eq. (11) summarises such “rate-of-return” effects, the negative tendency for savings being stronger than the positive one for $\gamma > 1$.

Eq. (11) also signals that the effects related to the increase in the expected marginal product of capital tend to be stronger than the ones induced by the mean preserving spread in the return of investment (the first addendum in eq. (11) is multiplied by $h\bar{p}^\varepsilon (1 + \bar{\theta})/[1 - \delta + h\bar{p}^\varepsilon + [h\bar{p}^\varepsilon - (r + \delta)]\bar{\theta}]$, which is lesser than 1, for $\bar{\theta} \in (-1, (1 - \delta)/(r + \delta))$). This happens because the increase in expected returns affects the first and second derivatives of the maximum value function (4), while a mean preserving spread in returns influences only its second and third derivatives (see Sandmo (1970) for a good explanation of the latter point in a two-period environment).

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13 In this set-up, Nakamura’s result holds only if $\delta$ is equal to one and $\bar{\theta}$ is zero or if $\bar{\theta}$ tends to infinity. In fact, recalling that $\varepsilon = 1/(1 - \alpha)$, in those cases $\text{Sign}\{\partial z_l/\partial \sigma^2_l\} = \text{Sign}\{(1 - \bar{\gamma})(\alpha - \bar{\gamma})\}$. Thus, as suggested by Nakamura (1999), the response of investment to an increase in future price volatility would be negative when the degree of risk aversion is lower than unity but higher than the labour productivity parameter in the Cobb-Douglas production function.
3. Optimal leverage: numerical results

The above analysis qualifies the optimal response of net investment to future price volatility given the level of the firm’s debt. However, to characterise the situation in which firms optimally choose their debt level, we must simultaneously solve eqs. (5) and (9). This system proves to be complex enough to justify the use of numerical techniques for its solution. Hence, we assume \( p_t \) to be log-normally distributed because this seems to be a reasonable compromise between realism and numerical manageability.14

In what follows, we choose \( \alpha = 0.66 \), which is roughly compatible with the observed labour income share, \( \delta = 0.1 \) and \( \beta = 0.98 \) (i.e. we “calibrate” our model using a time period equal to one year. The riskless rate is fixed at 0.05, while \( \bar{p} \) is normalised to unity. The nominal wage has been chosen to obtain values for \( h \) that imply “realistic” leverage levels. In other words, \( w \) is fixed at a level such that, given \( \alpha \), \( h \) generates an expected marginal revenue product (slightly) higher than the user’s cost of capital (which is \( r + \delta \)). The wedge between the latter and the expected marginal revenue represents the premium necessary to induce a risk-averse firm to contract some debt. The standard deviation for the idiosyncratic disturbance affecting the output price ranges from 0.015 to 0.045. Guiso and Parigi (1999) have inspired the choice for this interval. Their data are focused on “horizontal shifts” in demand (i.e., in changes in the quantity demanded while holding the price constant) and allow computing of the coefficient of variation for such shifts. The reported coefficient of the “one-period ahead” (subjective) distribution is 0.023. We can show that, with isoelastic demand functions and perfect competition, the coefficient of variation corresponds to our standard deviation.

Figure 1 studies the Cobb-Douglas case for a risk aversion index ranging from 1 to 6. Since the behaviour of “optimal leverage” as a function of \( \sigma_p \) does not differ appreciably as \( \gamma \) changes, in the first panel we choose to depict the “typical” optimal debt to internal resources ratio, drawn for \( \gamma = 3 \). The second panel shows the

14 Our routines have been developed in Gauss; the expectational integrals have been solved using the 10-points Gauss-Hermite quadrature procedure described in Judd (1998).
equilibrium net investment when the leverage is optimally chosen. To favour a comparison, for each level of risk-aversion the net investment performed when $\sigma_p = 0.015$ has been normalised to unity. When $\gamma > 4$, investment is decreasing in the standard deviation of the output price. As already hinted in the Introduction, we consider the equilibrium investment—in other words, the investment ratio is obtained by computing, from eq. (9), an average $z_t$ in which the distributions for $p_t$ and $p_{t+1}$ are characterised by the same standard deviation. Hence, also $\hat{\theta}_{t-1}$ and $\hat{\theta}_t$ assume the same value. This choice tends to favour the attainment of a positive “investment-uncertainty” relationship—in the “short run” (i.e., for a given $\hat{\theta}_{t-1}$), the reduction in $\hat{\theta}_t$ induces a temporary negative response of investment to an increase in future price uncertainty (eq. (10)). We have ignored this temporary effect in our simulations.

What is at work here is a “portfolio shift.” In general, as the environment becomes more uncertain a risk-averse agent increases the share of his or her wealth invested in safe assets. In our framework, the firm reduces its “short position” in the risk-free asset and thus diminishes its debt level. However, the contraction in leverage reduces the expected returns per unit of “internal resources” because the expected marginal revenue product is higher than the user cost of capital. The reduction in expected returns tends to depress investment (see eq. (9)).

[Figure 1 about here]

The second panel in Figure 1 is also useful for underscoring the fact that the equilibrium investment is exposed to different forces. When $\gamma$ is relatively low—specifically, less than or equal to 4—the “precautionary savings effects” can prevail over the “rate-of-return effect.” In this case, firms’ owners are not “very risk averse,” and thus the (positive) precautionary saving effect of uncertainty on investment is weak (see panel 3). However, the negative effect on investment induced by a reduction in $\hat{\theta}_t$ is weaker the lower is $\gamma$ (from eq. (11), and we can show that $\frac{\partial z_t}{\partial \theta \partial \gamma} > 0$). Our simulations reveal that the positive effect may prevail when $\gamma$ is relatively low.

The third panel depicts the situation in which investment is entirely financed by internal resources (i.e., $\theta_t$ is arbitrarily set to zero). In this case, the “precautionary
savings” effect guarantees a positive investment uncertainty relation, which is steeper the higher is the risk aversion index.

4. Extensions

4.1 CES specification for the production function. As is well know, our basic Cobb-Douglas technology implies a unit elasticity of substitution between capital and labour. This value may seem too high to describe a “short-period” situation, and for that reason we adopt the following constant return-to-scale specification:

\[ y_t = G[(1 - B)L_t^\rho + BK_t^\rho]^{1/\rho}, \]

where \( \rho \) is the parameter characterising the constant elasticity of substitution (which is equal to \( 1/(1-\rho) \)); in the simulation, the constants \( B \) and \( G \) are chosen so that the labour share of income is two thirds and the capital/output ratio is 2.

The solution for the intertemporal problem now takes the form:

\[ J(p_t, F_{t+1}) = A \left[ 1 - \delta - (r + \delta)\theta_{t+1} + h'(p_t)(1+\theta_{t+1}) \right]^{1-\gamma} \frac{F_{t+1}^{1-\gamma}}{(1-\gamma)}, \]

where \( h' = wB^{1/\rho}(1-B), f(p_t) = \left\{ [w/(p_tG(1-B))]^{\nu} + B - 1 \right\}^{-1/\nu} \) and \( \nu = \rho/(1-\rho) \). The under-determined coefficient \( A \) proves to be:

\[ A = \left\{ 1 - \frac{1}{\beta E_t[1 - \delta - (r + \delta)\hat{\delta}_t + h'(p_{t+1})(1+\hat{\theta}_t)]^{1-\gamma} \}^{-1/\gamma} \right\}^{\gamma}. \]

\( \hat{\theta}_t \) is the optimal debt-to-internal capital ratio, which is obtained, as before, by means of numerical techniques. The three panels in Figure 2 present the equilibrium investment-uncertainty relationship—for a risk-aversion parameter ranging from 1 to 3—when the elasticity of substitution is equal to 0.75, 0.50 and 0.25, respectively (i.e., when \( \rho \) is equal to \(-3.0, -1, \) and \(-3\)). In the first panel, the elasticity of substitution is relatively high, and the investment-uncertainty relationship is never monotonic decreasing. In the intermediate case, the relationship is decreasing when \( \gamma \) is (slightly) larger than 2, while in the case that the short run technology is “rigid” (third panel) an increase in price uncertainty negatively affects investment for a \( \gamma \) slightly larger than 1.
The intuition for this result is clear: the lower the elasticity of substitution, the less convex the function for the marginal revenue of capital.

Further calculations show that, if we set the relative risk-aversion parameter to 1.5, the investment-uncertainty relation is monotonic decreasing in the interval $\sigma_p \in [0.015, 0.045]$ when the elasticity of substitution between capital and labour is lower than 0.361. Notice that this value is compatible with many available estimates for the CES production function (see, e.g., Hamersmesh (1993) for an extensive, even if not up to date, survey of empirical contributions on this issue).

4.2 *A three-factors production function.* According to the standard model, the labour input can be decided upon once the level of the output price is known. This is a debatable assumption. Legal restraints to firings and union activities may heavily discourage the temporary employment of new workers during a “boom.” The issue is more relevant for firms that employ skilled labour because it usually takes time to find workers with the proper qualifications and to introduce them into the production process.

Thus, we now adopt a Cobb-Douglas technology with three factors of production:

$$y_t = L_t^{-\alpha} R_t^{\mu} K_t^{1-\alpha}. $$

The productive factor $R_t$ can be interpreted as raw materials, energy, semi-manufactured inputs and/or unskilled labour, given the Cobb-Douglas specification for

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15 This choice implies that a firm’s growth rate, which is equal to $z_t$, on the order of 2% a year, is a realistic value.

16 Hartman (1976) showed that greater uncertainty can decrease (increase) investment if the elasticity of substitution between capital and labour is sufficiently high (low). This result is driven by the assumption of decreasing returns to scale that characterises that paper.
technology, and $\mu$ represents the share of value added pertaining to the “flexible factors.” Although the firm decides upon the level of $R_t$ once it has observed the shock, the skilled labour is now considered as a “quasi-fixed” factor because its level must be contracted upon at the end of the previous period.

The maximum value function for the owners of the representative firm is now written as:

$$J(p_t, F_{t-1}) = \max \{z_t, \ell_t, \theta_t\} \{u(\hat{x}_t) + \beta E_t \left[ J(p_{t+1}, F_t) \right] \},$$

where $\ell_t$ denotes the labour-to-capital ratio ($L_t/K_t$), which must be determined optimally. The solution for the above problem takes the form:

$$J(p_t, F_{t-1}) = A \left[ 1 - \delta - (r + \delta)\theta_{t-1} + (h^* p_t^e \theta_{t-1}^{(\alpha - \mu)(1 - \mu)} - w \ell_{t-1})(1 + \theta_{t-1}) \right]^{(1 - \gamma)} F_{t-1}^{(1 - \gamma)} \left( 1 - \gamma \right),$$

where $\epsilon' = 1/(1 - \mu)$ and, $h^* = (1 - \mu)/(\mu / e)^{\mu e'}$, $e$ being the (time invariant) price for the flexible productive factor. The difficulty here is that $\hat{\ell}_t$ and $\hat{\theta}_t$ must be determined simultaneously, but this can be accomplished, as before, by means of numerical techniques. Some calculations yield:

$$A = \left\{ \frac{1}{\beta E_t \left[ 1 - \delta - (r + \delta)\theta_t + (h^* p_t^e \theta_t^{(\alpha - \mu)(1 - \mu)} - w \ell_t)(1 + \theta_t) \right]^{(1 - \gamma)} \left( 1 - \gamma \right)^{-1/\gamma} } \right\}^{-\gamma}.$$

To draw the three panels in Figure 3, we chose $\mu$ equal to 0.22, 0.33, and 0.44, respectively. These values imply that the share of the value added pertaining to “flexible” factors is equal to one third, one half, or two thirds of the income accruing to non-capital productive factors. As before, the risk-aversion parameter ranges from 1 to 3. Our calculations show that the investment-uncertainty relationship is always monotonic decreasing, except for the most unfavourable case (when $\mu = 0.44$), in which it is monotonic decreasing when $\gamma > 1.67$.

The intuition for this result is, again, simple: the lower the value of $\mu$, the less convex is the schedule for the marginal revenue product.
5. Concluding remarks

While we do not deny the importance of monopolistic power and of irreversibility in explaining the negative investment-uncertainty relationship, we believe we have cast new light on an alternative (but possibly complementary) channel shaping such relations. For realistic parameters values, we have shown that the reduction in external finance caused by an increase in uncertainty and the related decrease in returns per unit of internal resources, lead to a reduction in investment. This helps to explain the negative link between investment and uncertainty that emerges from the behaviour of firms characterised by a low degree of market power and by a negligible irreversibility problem in their investment activity.

Clearly, our framework may be further extended in various directions. For example, a heavier use of numerical techniques would allow the discussion of more complex stochastic structures for the output price. This could be accomplished, for example, by using the “collocation technique” that was well expounded in Judd (1998).

The fact that our framework abstracts from an adjustment cost function for capital is probably more restrictive than using an i.i.d. process. More precisely, we have implicitly assumed, as have Hartman (1972), Zeira (1990), Nakamura (1999) and others, that the adjustment costs for capital are prohibitively high within the period and negligible among different periods. However, as long as the marginal revenue product of capital is convex in the output price, the presence of adjustment costs does not change the sign of the investment-uncertainty relationship, as many important papers have shown under various alternative assumptions (see Abel (1983), Caballero (1991), and others).

The most limiting feature of our contribution lies in the analytical difficulty of relating the demand for external finance with the credit market conditions. This is due to the dynamic stochastic nature of our partial equilibrium framework. In particular, imperfections in credit markets may adversely affect the analysis developed in the present contribution: if firms are permanently credit constrained, the relationship between uncertainty and debt is seriously undermined.

However, some empirical studies suggest that the number of credit-rationed firms is relatively modest (to quote a figure from an already cited contribution, in the
Guiso and Parigi (1999) sample the share of constrained firms is 10.81%). Moreover, credit constraints should be more likely to be binding for firms whose investment is (almost) irreversible.

Finally, we wish to underscore that our model delivers some clearly testable implications, including a negative relationship between leverage ratio and perceived uncertainty and a positive relationship between net investment and debt. The econometric test of these predictions is left for future research.

Appendix

Proof of Proposition 1.

Consider first the derivative with respect to $\theta_t$ of the time-$t$ expected value for the tentative solution (4):

$$E_t \{ [(1 - \delta + h_{t+1}^r + (h_{t+1}^c - (r + \delta))\theta_t)^{-\gamma} [ h_{t+1}^c - (r + \delta) ] \}.$$

(A1)

When $\theta_t = -1$, the above expression becomes $E_t \{(1 + r)^{-\gamma} [ h_{t+1}^c - (r + \delta) ]\}$ which is positive by Assumption 2. Notice that:

$$\frac{\partial E_t \{[(1 - \delta + h_{t+1}^c + (h_{t+1}^c - (r + \delta))\theta_t)^{-\gamma} [ h_{t+1}^c - (r + \delta) ]\}}{\partial \theta_t} = -\gamma E_t \{[(1 - \delta + h_{t+1}^c + (h_{t+1}^c - (r + \delta))\theta_t)^{-(1+\gamma)} [ h_{t+1}^c - (r + \delta) ]^2 \} < 0$$

In fact, the expectation term is positive: since $\gamma \geq 1$ (Assumption 1), it can be seen as the second moment for the random variable:

$$\frac{h_{t+1}^c - (r + \delta)}{[1 - \delta + h_{t+1}^c + (h_{t+1}^c - (r + \delta))\theta_t]^{(1+\gamma)/2}}.$$

Now choose: $\theta_t = (1 - \delta)(r + \delta)$; expression (A1) becomes:

$$\left(\frac{1+r}{r+\delta}\right)^{-\gamma} E_t \{ (h_{t+1}^c)^{-(1+\gamma)} [ h_{t+1}^c - (r + \delta) ]\},$$

which is:
\[
\left(1 + \frac{r}{r + \delta}\right)^{-\gamma} \{ E_t \left[ (hp_{t+1}^\epsilon)^{1-\gamma} \right] - E_t \left[ (hp_{t+1}^\epsilon)^{-\gamma} \right] (r + \delta) \}.
\]

Since \( \partial \{ E_t \left[ (hp_{t+1}^\epsilon)^{1-\gamma} \right] - E_t \left[ (hp_{t+1}^\epsilon)^{-\gamma} \right] (r + \delta) \} / \partial r < 0 \), Assumption 2 guarantees that the above expression is negative. Thus, the proof is completed. \( \Box \)

Proof of Proposition 2.

Notice that the necessary and sufficient condition for \( A \) to be positive is \( \beta E_t[\psi_{t+1}^{1-\gamma}] < 1 \).

When Assumption 1 is satisfied, for \( \theta_i = -1 \) we have that \( E_t[\psi_{t+1}^{1-\gamma}] = (1 + r)^{1-\gamma} \leq 1 \).

Now consider:

\[
\frac{\partial E_t[\psi_{t+1}^{1-\gamma}]}{\partial \theta_i} = (1 - \gamma)E_t \{ [1 - \delta + hp_{t+1}^\epsilon + (hp_{t+1}^\epsilon - (r + \delta))\theta_i]^{-\gamma} [hp_{t+1}^\epsilon - (r + \delta)] \}.
\]

Because the proof of Proposition 1 implies that \( E_t \{ [1 - \delta + hp_{t+1}^\epsilon + (hp_{t+1}^\epsilon - (r + \delta))\theta_i]^{-\gamma} [hp_{t+1}^\epsilon - (r + \delta)] \} \geq 0 \), then

\[
\frac{\partial E_t[\psi_{t+1}^{1-\gamma}]}{\partial \theta_i} \leq 0.
\]

Hence, \( E_t[\psi_{t+1}^{1-\gamma}] \leq 1 \), and, since \( \beta < 1 \), \( \beta E_t[\psi_{t+1}^{1-\gamma}] < 1 \). \( \Box \)

Proof of Proposition 3.

Following Stokey and Lucas (1989), we prove this result by showing that:

\[
\lim_{T \to \infty} \beta^T E_t[J(p_{t,T}, F_{t,T-1})] = 0,
\]

where \( E_t[J(p_{t,T}, F_{t,T-1})] \) is the time \( t \) expected value for the time \( t + T \) maximum value function. That is:

\[
E_t[J(p_{t,T}, F_{t,T-1})] = E_t \left[ A(1 - \delta + hp_{t,T}^\epsilon + [hp_{t,T}^\epsilon - (r + \delta)\theta_{t,T-1}]^{-\gamma} F_{t,T-1}^{1-\gamma} \right].
\]

Exploiting equations (2) and (9), we can write:
\[
\lim_{T \to \infty} \beta^T E_t [J(p_{t,T}, F_{t+T-1})] = \lim_{T \to \infty} \beta^T A \frac{F_{t-1}^{1-\gamma}}{1-\gamma} E_t \left[ \prod_{j=0}^{T-1} (1 + z_{t+j})^{1-\gamma} \right] = \\
= \lim_{T \to \infty} \beta^T A \frac{F_{t-1}^{1-\gamma}}{1-\gamma} E_t \left[ \psi_{t+T}^{1-\gamma} \prod_{j=0}^{T-1} \left( \frac{\psi_{t+j}}{\beta E_t [\psi_{t+j+1}^{1-\gamma}]} \right)^{1-\gamma} \right]
\]

which is:

\[
\lim_{T \to \infty} \beta^T A \frac{F_{t-1}^{1-\gamma}}{1-\gamma} \psi_{t+T}^{1-\gamma} E_t \left[ \prod_{j=1}^{T} \frac{\psi_{t+j}}{\beta E_t [\psi_{t+j+1}^{1-\gamma}]} \right]^{(\gamma-1)/\gamma}.
\]

Because the terms \( \psi_{t+j} \) are independent, we transform the above expression into:

\[
\lim_{T \to \infty} \beta^T A \frac{F_{t-1}^{1-\gamma}}{1-\gamma} \psi_{t+T}^{1-\gamma} E_t \left[ \prod_{j=1}^{T} \frac{E_t [\psi_{t+j}^{1-\gamma}]}{\beta E_t [\psi_{t+j+1}^{1-\gamma}]} \right]^{(\gamma-1)/\gamma},
\]

and, hence,

\[
\lim_{T \to \infty} A \frac{F_{t-1}^{1-\gamma}}{1-\gamma} \psi_{t+T}^{1-\gamma} \left\{ \prod_{j=1}^{T} \left[ \beta E_t [\psi_{t+j}^{1-\gamma}] \right]^{1/\gamma} \right\}.
\]

Because \( \beta E_t [\psi_{t+j}^{1-\gamma}] = \beta E_t [\psi_{t+j}^{1-\gamma}] < 1 \forall j \), and \( \gamma > 1 \), the expression in the big square brackets converges to zero when \( T \) diverges. \( \mathbf{\varepsilon} \)

References


Figure 1. Two factors Cobb-Douglas tecnology

Figure 2. Net investment ratio vs. $\sigma_p$

Figure 3. Net investment ratio vs. $\sigma_p$ for different values of $\gamma$
Figure 2. CES between capital and labour
Figure 3. Three factors Cobb-Douglas technology